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Existence of solutions for some implicit partial differential equations and applications to variational integrals involving quasi-affine functions

Bernard Dacorogna and **Ana Margarida Ribeiro**
Institut de mathématiques, EPFL, 1015 Lausanne, Switzerland
(bernard.dacorogna@epfl.ch; ana.ribeiro@epfl.ch)

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We discuss some existence theorems for partial differential inclusions, subject to Dirichlet boundary conditions, of the form

$$\Phi(Du(x)) \in \{\alpha, \beta\} \quad \text{a.e. } x \in \Omega,$$

where Φ is a quasi-affine function and so, in particular, for $\Phi(Du) = \det Du$.

We then apply it to minimization problems of the form

$$\inf\left\{\int_\Omega g(\Phi(Du(x)))\,\mathrm{d}x : u \in \varphi + W_0^{1,\infty}(\Omega;\mathbb{R}^m)\right\}.$$

1. Introduction

In this article we discuss the existence of solutions for some first-order partial differential equations and then apply these results to minimization problems of the calculus of variations.

Let us first discuss the model case and introduce some notation (we will always adopt those of [5]). For maps $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$, we denote its gradient by $Du \in \mathbb{R}^{n\times n}$ and its determinant by $\det Du$.

We also, given a matrix $\xi \in \mathbb{R}^{n\times n}$, define the singular values of ξ as the eigenvalues of $(\xi\xi^{\mathrm{T}})^{1/2}$ and we denote them by

$$0 \leqslant \lambda_1(\xi) \leqslant \lambda_2(\xi) \leqslant \cdots \leqslant \lambda_n(\xi).$$

Our first theorem is the following.

THEOREM 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $\alpha < \beta$ and $0 < \gamma_2 \leqslant \cdots \leqslant \gamma_n$ be such that*

$$\gamma_2 \prod_{i=2}^n \gamma_i > \max\{|\alpha|, |\beta|\}.$$

Let $\varphi \in C^1_{\mathrm{piec}}(\bar{\Omega};\mathbb{R}^n)$ (the set of piecewise C^1 maps) be such that, for almost every $x \in \Omega$,

$$\alpha < \det D\varphi(x) < \beta,$$

$$\prod_{i=\nu}^n \lambda_i(D\varphi(x)) < \prod_{i=\nu}^n \gamma_i, \quad \nu = 2, \dots, n.$$

Then there exists $u \in \varphi + W_0^{1,\infty}(\Omega;\mathbb{R}^n)$ *such that*

$$\det Du \in \{\alpha, \beta\} \quad \text{a.e. in } \Omega,$$
$$\lambda_\nu(Du) = \gamma_\nu, \quad \nu = 2, \dots, n, \quad \text{a.e. in } \Omega.$$

REMARK 1.2. This theorem generalizes a theorem of Dacorogna and Marcellini [5] where $\beta = -\alpha > 0$.

REMARK 1.3. The theorem is also true if $\alpha = \beta \neq 0$ (the condition $\alpha < \det D\varphi < \beta$ being replaced by $\det D\varphi = \alpha$), and therefore also generalizes a theorem of Dacorogna and Tanteri [9].

We then apply this theorem (for details, see theorem 5.1) to the following minimization problem:

$$\inf\left\{\int_\Omega g(\det Du(x))\,\mathrm{d}x : u \in \varphi + W_0^{1,\infty}(\Omega;\mathbb{R}^n)\right\}. \tag{P}$$

This problem is important for applications (see [2] and [3]).

It should immediately be pointed out that, even when g is convex, it is not clear that (P) admits a minimizer (unless φ is affine, in which case $u = \varphi$ is a minimizer). It was proved in [2], and then extended in [6], that if Ω is smooth and φ is a $C^{1,\alpha}$, $0 < \alpha < 1$, diffeomorphism, then there exists a minimizer $\bar{u}$ of (P) that also solves

$$\det D\bar{u} = \frac{1}{|\Omega|}\int_\Omega \det D\varphi(y)\,\mathrm{d}y \quad \text{in } \Omega,$$
$$\bar{u} = \varphi \quad \text{on } \partial\Omega.$$

The non-convex case was then investigated by Mascolo and Schianchi [10] for non-affine φ and by Cellina and Zagatti [1] and Dacorogna and Marcellini [4] when φ is affine. Theorem 1.1 allows us to give a new proof of the existence of minimizers for (P) when g is non-convex.

We then discuss the case of quasi-affine functions. We recall that, for $m = n = 2$ (for the general case, $m, n \geqslant 2$ (see § 2)), a quasi-affine function is of the form

$$\Phi(\xi) = \Phi(0) + \langle \mu_1; \xi\rangle + \mu_2 \det \xi,$$

where $\mu_1 \in \mathbb{R}^{2\times 2}$ and $\mu_2 \in \mathbb{R}$.

We will then prove the following theorem, which is, in some aspects, more general than theorem 1.1 (since we can allow general quasi-affine functions) and, in others, weaker (since we cannot prescribe other equations such as $\lambda_i(Du) = \gamma_i$ (for some extensions, see [11])).

THEOREM 1.4. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded open set,* $\alpha < \beta$, $\Phi : \mathbb{R}^{m\times n} \to \mathbb{R}$ *a non-constant quasi-affine function and* $\varphi \in C^1_{\mathrm{piec}}(\bar{\Omega};\mathbb{R}^m)$ *such that, for almost every* $x \in \Omega$,

$$\alpha < \Phi(D\varphi(x)) < \beta.$$

Then there exists $u \in \varphi + W_0^{1,\infty}(\Omega;\mathbb{R}^m)$ *satisfying*

$$\Phi(Du) \in \{\alpha, \beta\} \quad \text{a.e. in } \Omega.$$

This theorem has a direct application to the minimization problem

$$\inf\left\{\int_{\Omega} g(\Phi(Du(x)))\,\mathrm{d}x : u \in \varphi + W_0^{1,\infty}(\Omega;\mathbb{R}^m)\right\}$$

when g is non-convex, recovering a theorem already proved, by different means, by Cellina and Zagatti [1].

2. Preliminaries

In this section we state the main abstract existence theorem that we will use in the following sections. We also briefly define the notion of a quasi-affine function.

We start by recalling the notion of a *rank-one convex hull* of a given set (for more details, see [5]).

NOTATION 2.1. For $E \subset \mathbb{R}^{m\times n}$, let

$$\begin{aligned}\bar{\mathcal{F}}_E &= \{f : \mathbb{R}^{m\times n} \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} \text{ and } f|_E \leqslant 0\},\\ \operatorname{Rco} E &= \{\xi \in \mathbb{R}^{m\times n} : f(\xi) \leqslant 0 \text{ for every rank-one convex } f \in \bar{\mathcal{F}}_E\}.\end{aligned}$$

We denote by $\operatorname{Int}\operatorname{Rco} E$ the interior of the rank-one convex hull of E.

We start with the following definition introduced by Dacorogna and Marcellini in [5], which is the key condition to get the existence of solutions.

DEFINITION 2.2 (approximation property). Let $E \subset K(E) \subset \mathbb{R}^{m\times n}$. The sets E and $K(E)$ are said to have the *approximation property* if there exists a family of closed sets E_δ and $K(E_\delta)$, $\delta > 0$, such that the following hold.

(1) $E_\delta \subset K(E_\delta) \subset \operatorname{Int} K(E)$ for every $\delta > 0$.

(2) For every $\varepsilon > 0$, there exists $\delta_0 = \delta_0(\varepsilon) > 0$ such that $\operatorname{dist}(\eta; E) \leqslant \varepsilon$ for every $\eta \in E_\delta$ and $\delta \in [0, \delta_0]$.

(3) If $\eta \in \operatorname{Int} K(E)$, then $\eta \in K(E_\delta)$ for every $\delta > 0$ sufficiently small.

The main abstract existence theorem that we use in our analysis is as follows (cf. theorem 6.3 combined with theorem 6.14 in [5], or, for a slightly more general version, that we use here, cf. theorem 7 in [7]).

THEOREM 2.3. *Let $\Omega \subset \mathbb{R}^n$ be open. Let $E \subset \mathbb{R}^{m\times n}$ be compact. Assume that* $\operatorname{Rco} E$ *has the approximation property with $K(E_\delta) = \operatorname{Rco} E_\delta$. Let $\varphi \in C^1_{\mathrm{piec}}(\bar{\Omega};\mathbb{R}^m)$ (where C^1_{piec} denotes the set of piecewise C^1 maps) be such that*

$$D\varphi(x) \in E \cup \operatorname{Int}\operatorname{Rco} E \quad \text{a.e. in } \Omega.$$

Then there exists (a dense set of) $u \in \varphi + W_0^{1,\infty}(\Omega;\mathbb{R}^m)$ such that

$$Du(x) \in E \quad \text{a.e. in } \Omega.$$

Finally, we recall the notion of *quasi-affine functions* (for more details, see [3]).

DEFINITION 2.4. We say that $\Phi : \mathbb{R}^{m\times n} \to \mathbb{R}$ is quasi-affine if

$$\Phi(\xi) = \Phi(0) + \sum_{k=1}^{m\wedge n} \langle A^k; \operatorname{adj}_k \xi \rangle,$$

where $m \wedge n = \min\{n, m\}$, $A^k \in \mathbb{R}^{\sigma(k)}$, $\sigma(k) = \binom{m}{k} \times \binom{n}{k}$, $\operatorname{adj}_k \xi$ is the matrix of the minors of ξ of order k and $\langle \cdot; \cdot \rangle$ denotes the scalar product.

In an equivalent form, we can write

$$\Phi(\xi) = \Phi(0) + \sum_{q=1}^{m\wedge n} \sum_{\substack{1\leqslant i_1<\cdots<i_q\leqslant m \\ 1\leqslant j_1<\cdots<j_q\leqslant n}} \mu^{i_1\cdots i_q}_{j_1\cdots j_q} \det \begin{pmatrix} \xi^{i_1}_{j_1} & \cdots & \xi^{i_1}_{j_q} \\ \vdots & & \vdots \\ \xi^{i_q}_{j_1} & \cdots & \xi^{i_q}_{j_q} \end{pmatrix}$$

for some constants $\mu^{i_1\cdots i_q}_{j_1\cdots j_q} \in \mathbb{R}$, $1 \leqslant q \leqslant m \wedge n$.

Moreover, we have the following result.

PROPOSITION 2.5. *Let $\Phi : \mathbb{R}^{m\times n} \to \mathbb{R}$ be quasi-affine and $\Omega \subset \mathbb{R}^n$ be a bounded open set. Then*

$$\int_\Omega \Phi(Dv(x))\,\mathrm{d}x = \int_\Omega \Phi(Du(x))\,\mathrm{d}x \quad \forall v \in u + W^{1,\infty}_0(\Omega; \mathbb{R}^m).$$

3. Rank-one convex hulls

In this section we compute the rank-one convex hull of sets E involving the condition

$$\Phi(\xi) \in \{\alpha, \beta\},$$

where Φ is a quasi-affine function. We start in § 3.1 with the case of the determinant where extra conditions on the singular values are allowed. In § 3.2 we deal with general quasi-affine functions.

3.1. The case of the determinant

We prove the following theorem.

THEOREM 3.1. *Let $\alpha \leqslant \beta$, $0 < \gamma_2 \leqslant \cdots \leqslant \gamma_n$ be constants such that*

$$\gamma_2 \prod_{i=2}^{n} \gamma_i \geqslant \max\{|\alpha|, |\beta|\}.$$

Let

$$E = \{\xi \in \mathbb{R}^{n\times n} : \det \xi \in \{\alpha, \beta\},\ \lambda_i(\xi) = \gamma_i,\ i = 2, \dots, n\}.$$

Then

$$\operatorname{Rco} E = \left\{\xi \in \mathbb{R}^{n\times n} : \det \xi \in [\alpha, \beta],\ \prod_{i=\nu}^{n} \lambda_i(\xi) \leqslant \prod_{i=\nu}^{n} \gamma_i,\ \nu = 2, \dots, n\right\}.$$

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Moreover, if $\alpha < \beta$*, then*

$$\operatorname{Int}\operatorname{Rco} E = \left\{\xi \in \mathbb{R}^{n\times n} : \det \xi \in (\alpha, \beta),\ \prod_{i=\nu}^{n} \lambda_i(\xi) < \prod_{i=\nu}^{n} \gamma_i,\ \nu = 2, \dots, n\right\},$$

and if $\alpha = \beta$*, then*

$$\operatorname{Int}\operatorname{Rco} E = \left\{\xi \in \mathbb{R}^{n\times n} : \det \xi = \alpha,\ \prod_{i=\nu}^{n} \lambda_i(\xi) < \prod_{i=\nu}^{n} \gamma_i,\ \nu = 2, \dots, n\right\},$$

where the interior is to be understood relative to the manifold $\{\det \xi = \alpha\}$*.*

Remark 3.2. The theorem extends [8] and [5] if $\beta = -\alpha > 0$ and [9] if $\alpha = \beta$. In particular, note that if we let, when $\beta = -\alpha > 0$,

$$\gamma_1 = \beta \left(\prod_{i=2}^{n} \gamma_i\right)^{-1},$$

then

$$\begin{aligned} E &= \{\xi \in \mathbb{R}^{n\times n} : \det \xi \in \{-\beta, \beta\},\ \lambda_i(\xi) = \gamma_i,\ i = 2, \dots, n\} \\ &= \{\xi \in \mathbb{R}^{n\times n} : \lambda_1(\xi) = \gamma_1,\ \lambda_i(\xi) = \gamma_i,\ i = 2, \dots, n\}. \end{aligned}$$

Proof. We divide the proof into two parts. In the first one, we obtain the characterization of $\operatorname{Rco} E$, and in the second a characterization of its interior.

Part 1. Let

$$X = \left\{\xi \in \mathbb{R}^{n\times n} : \det \xi \in [\alpha, \beta],\ \prod_{i=\nu}^{n} \lambda_i(\xi) \leqslant \prod_{i=\nu}^{n} \gamma_i,\ \nu = 2, \dots, n\right\}.$$

We want to show that $X = \operatorname{Rco} E$.

Step 1 ($\operatorname{Rco} E \subset X$). This is the easy implication. Indeed, observe that $E \subset X$ and that the functions

$$\xi \to \pm \det \xi, \quad \xi \to \prod_{i=\nu}^{n} \lambda_i(\xi), \quad \nu = 2, \dots, n,$$

are rank-one convex (see [5]). We therefore have that the set X is rank-one convex and thus the desired inclusion.

Step 2 ($X \subset \operatorname{Rco} E$). Since the set X is compact (the function $\xi \to \lambda_n(\xi)$ being a norm), it is enough to show that $\partial X \subset \operatorname{Rco} E$. So we let $\xi \in \partial X$ and we want to prove that $\xi \in \operatorname{Rco} E$. Note that $\partial X = X_\alpha \cup X_\beta \cup X_2 \cup \dots \cup X_n$, where

$$\begin{aligned} X_\alpha &= \{\xi \in X : \det \xi = \alpha\}, \\ X_\beta &= \{\xi \in X : \det \xi = \beta\}, \\ X_\nu &= \left\{\xi \in X : \prod_{i=\nu}^{n} \lambda_i(\xi) = \prod_{i=\nu}^{n} \gamma_i\right\} \end{aligned}$$

for $\nu = 2, \dots, n$.

Since all the functions involved in the definition of X are right and left $SO(n)$ invariant, there is no loss of generality in assuming that ξ is diagonal,

$$\xi = \operatorname{diag}(x_1, x_2, \dots, x_n),$$

with $0 \leqslant |x_1| \leqslant x_2 \leqslant \cdots \leqslant x_n$. We therefore have $\lambda_1(\xi) = |x_1|$, $\lambda_i(\xi) = x_i$, $i = 2, \dots, n$. We will now proceed by induction on the dimension n; when $n = 1$ the result is trivial.

Several possibilities can then happen, bearing in mind that $\xi \in \partial X$.

Case 1. $\xi \in X_{\bar\nu}$ for a certain $\bar\nu = 2, \dots, n$, i.e.

$$\prod_{i=\bar\nu}^{n} x_i = \prod_{i=\bar\nu}^{n} \gamma_i.$$

We write $\xi \in \mathbb{R}^{n\times n}$ as two blocks, one in $\mathbb{R}^{(\bar\nu-1)\times(\bar\nu-1)}$ and one in $\mathbb{R}^{(n-\bar\nu+1)\times(n-\bar\nu+1)}$, in the following way: $\xi = \operatorname{diag}(\xi_{\bar\nu-1}, \xi_{n-\bar\nu+1})$, where $\xi_{\bar\nu-1} = \operatorname{diag}(x_1, \dots, x_{\bar\nu-1})$ and $\xi_{n-\bar\nu+1} = \operatorname{diag}(x_{\bar\nu}, \dots, x_n)$.

We then apply the hypothesis of induction on $\xi_{\bar\nu-1}$ and $\xi_{n-\bar\nu+1}$ (we will check that we can do so below) and we deduce that $\xi \in \operatorname{Rco} E$. Let us now see that we can apply the hypothesis of induction first for $\xi_{\bar\nu-1}$. We have (when $\bar\nu = 2$ or $\bar\nu = n$, terms such as $\prod_{i=2}^{\bar\nu-1}$ or $\prod_{i=\bar\nu+1}^{n}$ should be replaced by 1)

$$\gamma_2 \prod_{i=2}^{\bar\nu-1} \gamma_i = \gamma_2 \prod_{i=2}^{n} \gamma_i \biggl(\prod_{i=\bar\nu}^{n} \gamma_i\biggr)^{-1} \geqslant \max\biggl\{\frac{|\alpha|}{\gamma_{\bar\nu}\cdots\gamma_n}, \frac{|\beta|}{\gamma_{\bar\nu}\cdots\gamma_n}\biggr\},$$

$$\begin{aligned} \det \xi_{\bar\nu-1} &= \prod_{i=1}^{\bar\nu-1} x_i = \prod_{i=1}^{n} x_i \biggl(\prod_{i=\bar\nu}^{n} x_i\biggr)^{-1} \\ &= \prod_{i=1}^{n} x_i \biggl(\prod_{i=\bar\nu}^{n} \gamma_i\biggr)^{-1} = \det\xi \biggl(\prod_{i=\bar\nu}^{n} \gamma_i\biggr)^{-1} \in \biggl[\frac{\alpha}{\gamma_{\bar\nu}\cdots\gamma_n}, \frac{\beta}{\gamma_{\bar\nu}\cdots\gamma_n}\biggr], \end{aligned}$$

$$\prod_{i=\nu}^{\bar\nu-1} \lambda_i(\xi_{\bar\nu-1}) = \prod_{i=\nu}^{n} x_i \biggl(\prod_{i=\bar\nu}^{n} x_i\biggr)^{-1} = \prod_{i=\nu}^{n} x_i \biggl(\prod_{i=\bar\nu}^{n} \gamma_i\biggr)^{-1} \leqslant \prod_{i=\nu}^{\bar\nu-1} \gamma_i, \quad \nu = 2, \dots, \bar\nu - 1,$$

and thus the result.

Similarly, for $\xi_{n-\bar\nu+1}$, since (here, the role of α and β is played, for both, by $\prod_{i=\bar\nu}^{n} \gamma_i$)

$$\begin{aligned} \gamma_{\bar\nu+1} \prod_{i=\bar\nu+1}^{n} \gamma_i &\geqslant \prod_{i=\bar\nu}^{n} \gamma_i, \\ \det \xi_{n-\bar\nu+1} &= \prod_{i=\bar\nu}^{n} x_i = \prod_{i=\bar\nu}^{n} \gamma_i, \\ \prod_{i=\nu-\bar\nu+1}^{n-\bar\nu+1} \lambda_i(\xi_{n-\bar\nu+1}) &= \prod_{i=\nu}^{n} x_i \leqslant \prod_{i=\nu}^{n} \gamma_i, \quad \nu = \bar\nu + 1, \dots, n, \end{aligned}$$

we have the claim.

Case 2. $\xi \in X_\alpha$ (similarly for the case $\xi \in X_\beta$). We can also assume that $\xi \notin X_\nu$, $\nu = 2, \dots, n$, otherwise we apply case 1. So we can assume that

$$\xi \in \operatorname{Int} X_\alpha = \left\{ \eta \in \mathbb{R}^{n\times n} : \det \eta = \alpha,\ \prod_{i=\nu}^{n} \lambda_i(\eta) < \prod_{i=\nu}^{n} \gamma_i,\ \nu = 2, \dots, n \right\}.$$

This is clearly an open set (relative to the manifold $\{\det \eta = \alpha\}$).

Recall that

$$\xi = \operatorname{diag}(x_1, \dots, x_n) = \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix}.$$

We then set, for $t \in \mathbb{R}$,

$$\xi_t = \begin{pmatrix} x_1 & & & \\ & \ddots & & \\ & & x_{n-1} & t \\ & & 0 & x_n \end{pmatrix}$$

and observe that $\det \xi_t = \det \xi = \alpha$. Since X_α is bounded, we can find $t_1 < 0 < t_2$ such that $\xi_{t_1}, \xi_{t_2} \in \partial X_\alpha$, which means that $\xi_{t_i} \in X_{\nu_i}$, $i = 1, 2$, for a certain $\nu_i = 2, \dots, n$, and therefore, by case 1, we have $\xi_{t_i} \in \operatorname{Rco} E$, and thus, since $\operatorname{rank}(\xi_{t_1} - \xi_{t_2}) = 1$, we deduce that $\xi \in \operatorname{Rco} E$, as required.

This concludes the first part of the theorem.

Part 2. The representation formula for $\operatorname{Int}\operatorname{Rco} E$ is easy and its proof is very similar to the ones in [5] or [8] and we skip the details. □

3.2. The case of a quasi-affine function

We will need, prior to the main theorem, two elementary lemmas, but we postpone their proofs to the end of the present subsection. The first one will be used to assert that condition (3.1) below can be fulfilled by some $c_j^i > 0$ and will also be used in theorem 1.4. Lemma 3.4 will be used in the proof of theorem 3.5.

Lemma 3.3. *Let $\Phi : \mathbb{R}^{m\times n} \to \mathbb{R}$ be a non-constant quasi-affine function and $M, N > 0$. Then there exist $c_j^i > N$, $i = 1, \dots, m, j = 1, \dots, n$, such that*

$$\inf\{|\Phi(\xi)| : |\xi_j^i| = c_j^i\} > M.$$

Lemma 3.4. *Let $\Phi : \mathbb{R}^{m\times n} \to \mathbb{R}$ be a non-constant quasi-affine function. Then Φ has no local extremum.*

We can now state the main theorem.

Theorem 3.5. *Let $\Phi : \mathbb{R}^{m\times n} \to \mathbb{R}$ be a non-constant quasi-affine function, $\alpha < \beta$, $c_j^i > 0$ satisfying*

$$\inf\{|\Phi(\xi)| : |\xi_j^i| = c_j^i\} > \max\{|\alpha|, |\beta|\}. \tag{3.1}$$

Let

$$E = \{\xi \in \mathbb{R}^{m\times n} : \Phi(\xi) \in \{\alpha, \beta\},\ |\xi^i_j| \leqslant c^i_j,\ i = 1, \dots, m,\ j = 1, \dots, n\}.$$

Then

$$\begin{aligned}\operatorname{Rco} E &= \{\xi \in \mathbb{R}^{m\times n} : \Phi(\xi) \in [\alpha, \beta],\ |\xi^i_j| \leqslant c^i_j,\ i = 1, \dots, m,\ j = 1, \dots, n\},\\ \operatorname{Int}\operatorname{Rco} E &= \{\xi \in \mathbb{R}^{m\times n} : \Phi(\xi) \in (\alpha, \beta),\ |\xi^i_j| < c^i_j,\ i = 1, \dots, m,\ j = 1, \dots, n\}.\end{aligned}$$

Proof.

Part 1. We let

$$X = \{\xi \in \mathbb{R}^{m\times n} : \Phi(\xi) \in [\alpha, \beta],\ |\xi^i_j| \leqslant c^i_j,\ i = 1, \dots, m,\ j = 1, \dots, n\}$$

and we show that $X = \operatorname{Rco} E$. The inclusion $\operatorname{Rco} E \subset X$ follows from the combination of the facts that $E \subset X$ and that the set X is rank-one convex (the functions Φ, $-\Phi$ and $|\cdot|$ being rank-one convex).

We therefore have to show only that $X \subset \operatorname{Rco} E$. So we let $\xi \in X$ and we assume that $\alpha < \Phi(\xi) < \beta$, otherwise the result is trivial. We observe that (3.1) implies that, for every $\xi \in X$, there exists (i, j) such that $|\xi^i_j| < c^i_j$. So, for $t \in \mathbb{R}$, let

$$\xi^t = \xi + te^i \otimes e_j$$

and observe that, by compactness, there exist $t_1 < 0 < t_2$ such that $\xi^{t_\nu} \in \partial X$, $\nu = 1, 2$, which implies that either $\Phi(\xi^{t_\nu}) \in \{\alpha, \beta\}$ or $|(\xi^{t_\nu})^i_j| = c^i_j$, $\nu = 1, 2$. If the first possibility happens, then we are done. If, however, the second case holds, then we restart the process with a different (i, j), since, by (3.1), it is not possible that $|(\xi^{t_\nu})^i_j| = c^i_j$ for every (i, j).

Part 2. We now define

$$Y = \{\xi \in \mathbb{R}^{m\times n} : \Phi(\xi) \in (\alpha, \beta),\ |\xi^i_j| < c^i_j,\ i = 1, \dots, m,\ j = 1, \dots, n\}$$

and observe that, since $Y \subset \operatorname{Rco} E$ and Y is open, then $Y \subset \operatorname{Int}\operatorname{Rco} E$. So let us show the reverse inclusion and choose $\xi \in \operatorname{Int}\operatorname{Rco} E$. Clearly, such a ξ must have $|\xi^i_j| < c^i_j$. Lemma 3.4 shows also that ξ should be such that $\alpha < \Phi(\xi) < \beta$. These observations imply the result. $\square$

We now prove lemma 3.3.

Proof. Since Φ is quasi-affine, we can write

$$\Phi(\xi) = \Phi(0) + \sum_{q=1}^{m\wedge n} \sum_{\substack{1\leqslant i_1<\cdots<i_q\leqslant m\\ 1\leqslant j_1<\cdots<j_q\leqslant n}} \mu^{i_1\cdots i_q}_{j_1\cdots j_q} \det\begin{pmatrix}\xi^{i_1}_{j_1} & \cdots & \xi^{i_1}_{j_q}\\ \vdots & & \vdots\\ \xi^{i_q}_{j_1} & \cdots & \xi^{i_q}_{j_q}\end{pmatrix}.$$

Since Φ is not constant, we can find $1 \leqslant s \leqslant m \wedge n$, $1 \leqslant i_1 < \cdots < i_s \leqslant m$ and $1 \leqslant j_1 < \cdots < j_s \leqslant n$ such that

$$\mu^{i_1\cdots i_s}_{j_1\cdots j_s} \neq 0 \quad \text{and} \quad \mu^{i_1\cdots i_q}_{j_1\cdots j_q} = 0 \quad \forall q > s.$$

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Assume, without loss of generality, that

$$\mu_{1\cdots s}^{1\cdots s} \neq 0. \tag{3.2}$$

Let us define the set

$$\Theta = \{\theta \in \mathbb{R}^{m\times n} : \theta_j^i \in \{\pm 1\}\}$$

and the product $A \odot B \in \mathbb{R}^{m\times n}$, for two given matrices $A, B \in \mathbb{R}^{m\times n}$, as

$$(A \odot B)_j^i = A_j^i \cdot B_j^i.$$

We want to find a matrix $C \in \mathbb{R}^{m\times n}$ such that $c_j^i > N$ and

$$\xi = C \odot \theta, \quad \theta \in \Theta \quad \Rightarrow \quad |\Phi(\xi)| > M.$$

In fact, we will prove that the matrix can be chosen of the form $C = \tau A$, where $\tau > 0$ and, for $t > 0$,

$$A_j^i = \begin{cases} t & \text{if } 1 \leqslant i = j \leqslant s, \\ 1 & \text{otherwise (i.e. if } i \neq j \text{ or if } i = j \geqslant s+1). \end{cases}$$

We observe that

$$\begin{aligned} \Phi(\xi) &= \Phi(C \odot \theta) \\ &= \Phi(0) + \sum_{q=1}^{s} \tau^q \sum_{\substack{1 \leqslant i_1 < \cdots < i_q \leqslant m \\ 1 \leqslant j_1 < \cdots < j_q \leqslant n}} \mu_{j_1 \cdots j_q}^{i_1 \cdots i_q} \det \begin{pmatrix} A_{j_1}^{i_1}\theta_{j_1}^{i_1} & \cdots & A_{j_q}^{i_1}\theta_{j_q}^{i_1} \\ \vdots & & \vdots \\ A_{j_1}^{i_q}\theta_{j_1}^{i_q} & \cdots & A_{j_q}^{i_q}\theta_{j_q}^{i_q} \end{pmatrix}, \end{aligned}$$

and that, for τ and t sufficiently large, it is possible to find $\gamma > 0$ such that

$$|\Phi(\xi)| \geqslant \gamma \tau^s t^s.$$

Choosing τ and t sufficiently large, we have, indeed, found $c_j^i > N$ and $|\Phi(\xi)| > M$ as required. □

We now prove lemma 3.4.

Proof. We prove that if Φ has a local extremum, then it must be constant. We proceed in two steps.

Step 1. We first show that if ξ is a local extremum point of Φ, then Φ is constant in a neighbourhood of ξ.

Assume that ξ is a local minimum point of Φ (the case of a local maximizer being handled similarly). We therefore have that there exists $\varepsilon > 0$ such that

$$\Phi(\xi) \leqslant \Phi(\xi + v) \quad \text{for every } v \in \mathbb{R}^{m\times n} \text{ such that } |v_j^i| \leqslant \varepsilon. \tag{3.3}$$

We show that this implies that

$$\Phi(\xi) = \Phi(\xi + v) \quad \text{for every } v \in \mathbb{R}^{m\times n} \text{ such that } |v_j^i| \leqslant \varepsilon. \tag{3.4}$$

We write

$$v = \sum_{\substack{1 \leqslant i \leqslant m \\ 1 \leqslant j \leqslant n}} v_j^i e^i \otimes e_j$$

and observe that, since Φ is quasi-affine,

$$\Phi(\xi) = \tfrac{1}{2}\Phi(\xi + v_1^1 e^1 \otimes e_1) + \tfrac{1}{2}\Phi(\xi - v_1^1 e^1 \otimes e_1),$$

and since (3.3) is satisfied, we deduce that

$$\Phi(\xi \pm v_1^1 e^1 \otimes e_1) = \Phi(\xi), \quad |v_1^1| \leqslant \varepsilon. \tag{3.5}$$

We next write, using again the fact that Φ is quasi-affine,

$$\Phi(\xi + v_1^1 e^1 \otimes e_1) = \tfrac{1}{2}\Phi(\xi + v_1^1 e^1 \otimes e_1 + v_2^1 e^1 \otimes e_2) + \tfrac{1}{2}\Phi(\xi + v_1^1 e^1 \otimes e_1 - v_2^1 e^1 \otimes e_2),$$

and since (3.3) and (3.5) hold, we deduce that

$$\Phi(\xi + v_1^1 e^1 \otimes e_1 \pm v_2^1 e^1 \otimes e_2) = \Phi(\xi + v_1^1 e^1 \otimes e_1) = \Phi(\xi), \quad |v_1^1|, |v_2^1| \leqslant \varepsilon.$$

Iterating the procedure, we have indeed established (3.4).

Step 2. We now show that if Φ is locally constant around a point $\xi \in \mathbb{R}^{m\times n}$, then Φ is constant everywhere, establishing the result. So assume that

$$\Phi(\xi + v) = \Phi(\xi) \quad \forall v \in \mathbb{R}^{m\times n}, \quad \text{with } |v_j^i| \leqslant \varepsilon, \tag{3.6}$$

and let us show that

$$\Phi(\xi + w) = \Phi(\xi) \quad \forall w \in \mathbb{R}^{m\times n}. \tag{3.7}$$

The procedure is similar to that of step 1 and we start to show that, for all $w_1^1 \in \mathbb{R}$ and $|v_j^i| \leqslant \varepsilon$, we have

$$\Phi\biggl(\xi + w_1^1 e^1 \otimes e_1 + \sum_{(i,j)\neq(1,1)} v_j^i e^i \otimes e_j\biggr) = \Phi(\xi + w_1^1 e^1 \otimes e_1) = \Phi(\xi). \tag{3.8}$$

Indeed, if $|w_1^1| \leqslant \varepsilon$, this is nothing else than (3.6), so we may assume that $|w_1^1| > \varepsilon$ and use the fact that Φ is quasi-affine to deduce that

$$\begin{aligned}\Phi\biggl(\xi + \varepsilon\frac{w_1^1}{|w_1^1|} e^1 \otimes e_1 + \sum_{(i,j)\neq(1,1)} v_j^i e^i \otimes e_j\biggr) &= \frac{\varepsilon}{|w_1^1|}\Phi\biggl(\xi + w_1^1 e^1 \otimes e_1 + \sum_{(i,j)\neq(1,1)} v_j^i e^i \otimes e_j\biggr) \\ &\quad + \biggl(1 - \frac{\varepsilon}{|w_1^1|}\biggr)\Phi\biggl(\xi + \sum_{(i,j)\neq(1,1)} v_j^i e^i \otimes e_j\biggr).\end{aligned}$$

Therefore, appealing to (3.6) and to the preceding identity, we have indeed established (3.8). Proceeding iteratively in a similar manner with the other components $(w_2^1, w_3^1, \dots)$, we have obtained (3.7) and thus the proof of the lemma is complete. □

4. Existence of solutions

We discuss the proofs of the two main theorems of § 1.

4.1. The case of the determinant

We recall theorem 1.1.

THEOREM 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $\alpha < \beta$ and $0 < \gamma_2 \leqslant \cdots \leqslant \gamma_n$ be such that*

$$\gamma_2 \prod_{i=2}^{n} \gamma_i > \max\{|\alpha|, |\beta|\}.$$

Let $\varphi \in C^1_{\mathrm{piec}}(\bar{\Omega}; \mathbb{R}^n)$ (the set of piecewise C^1 maps) be such that, for almost every $x \in \Omega$,

$$\alpha < \det D\varphi(x) < \beta,$$
$$\prod_{i=\nu}^{n} \lambda_i(D\varphi(x)) < \prod_{i=\nu}^{n} \gamma_i, \quad \nu = 2, \dots, n.$$

Then there exists $u \in \varphi + W_0^{1,\infty}(\Omega; \mathbb{R}^n)$ such that

$$\det Du \in \{\alpha, \beta\} \quad \text{a.e. in } \Omega,$$
$$\lambda_\nu(Du) = \gamma_\nu, \quad \nu = 2, \dots, n, \quad \text{a.e. in } \Omega.$$

Proof. We now show that the result follows from the combination of theorems 2.3 and 3.1. From theorem 3.1, we have

$$E = \{\xi \in \mathbb{R}^{n\times n} : \det \xi \in \{\alpha, \beta\},\ \lambda_i(\xi) = \gamma_i,\ i = 2, \dots, n\},$$
$$\mathrm{Rco}\, E = \left\{\xi \in \mathbb{R}^{n\times n} : \det \xi \in [\alpha, \beta],\ \prod_{i=\nu}^{n} \lambda_i(\xi) \leqslant \prod_{i=\nu}^{n} \gamma_i,\ \nu = 2, \dots, n\right\}.$$

Since $\varphi \in C^1_{\mathrm{piec}}(\bar{\Omega}; \mathbb{R}^n)$ and $D\varphi \in \mathrm{Int}\, \mathrm{Rco}\, E$, we only need to verify that E and $\mathrm{Rco}\, E$ have the approximation property.

For $\delta > 0$ such that $\gamma_2 - \delta > 0$ and $\alpha + \delta < \beta - \delta$, let

$$E_\delta = \{\xi \in \mathbb{R}^{n\times n} : \det \xi \in \{\alpha + \delta, \beta - \delta\},\ \lambda_i(\xi) = \gamma_i - \delta,\ i = 2, \dots, n\}.$$

For a sufficiently small δ, we have

$$(\gamma_2 - \delta) \prod_{i=2}^{n} (\gamma_i - \delta) \geqslant \max\{|\alpha + \delta|, |\beta - \delta|\},$$

and thus theorem 3.1 ensures that

$$\mathrm{Rco}\, E_\delta = \left\{\xi \in \mathbb{R}^{n\times n} : \det \xi \in [\alpha + \delta, \beta - \delta],\ \prod_{i=\nu}^{n} \lambda_i(\xi) \leqslant \prod_{i=\nu}^{n} (\gamma_i - \delta),\ \nu = 2, \dots, n\right\}.$$

We have to verify the three conditions of definition 2.2. The first one is obvious. We next verify the second condition. Since $\eta \in E_\delta$, we assume that $\det \eta = \alpha + \delta$, the case $\det \eta = \beta - \delta$ being handled in an analogous way. The set E_δ being left and right $SO(n)$ invariant, we can assume that

$$\eta = \operatorname{diag}\left(\frac{\alpha+\delta}{(\gamma_2-\delta)\cdots(\gamma_n-\delta)}, \gamma_2-\delta, \dots, \gamma_n-\delta\right).$$

If we let

$$\xi = \operatorname{diag}\left(\frac{\alpha}{\gamma_2\cdots\gamma_n}, \gamma_2, \dots, \gamma_n\right),$$

we have $\xi \in E$ and

$$\operatorname{dist}(\eta; E) \leqslant \max\left\{\left|\frac{\alpha+\delta}{(\gamma_2-\delta)\cdots(\gamma_n-\delta)} - \frac{\alpha}{\gamma_2\cdots\gamma_n}\right|, \delta\right\} \to 0 \quad \text{as } \delta \to 0.$$

The second condition of definition 2.2 then follows.

The third condition of the approximation property follows from the continuity of the functions involved in the definition of $\operatorname{Rco} E_\delta$. We may then apply theorem 2.3 to get the result. □

4.2. The case of a quasi-affine function

We recall theorem 1.4.

THEOREM 1.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $\alpha < \beta$, $\Phi : \mathbb{R}^{m\times n} \to \mathbb{R}$ a non-constant quasi-affine function and $\varphi \in C^1_{\text{piec}}(\bar{\Omega}; \mathbb{R}^m)$ such that, for almost every $x \in \Omega$,*

$$\alpha < \Phi(D\varphi(x)) < \beta.$$

Then there exists $u \in \varphi + W^{1,\infty}_0(\Omega; \mathbb{R}^m)$ satisfying

$$\Phi(Du) \in \{\alpha, \beta\} \quad \textit{a.e. in } \Omega.$$

REMARK 4.1. The theorem is, in fact, slightly more precise and asserts also that if c^i_j, $i = 1, \dots, m$, $j = 1, \dots, n$, are constants such that $|D_j\varphi^i(x)| < c^i_j$ and

$$|\Phi(\xi)| > \max\{|\alpha|, |\beta|\} \quad \forall \xi \in \mathbb{R}^{m\times n}, \quad |\xi^i_j| = c^i_j, \quad i = 1, \dots, m, \quad j = 1, \dots, n,$$

then the solutions also verify

$$|D_j u^i(x)| \leqslant c^i_j \quad \forall (i, j).$$

Proof. As $\varphi \in C^1_{\text{piec}}(\bar{\Omega}; \mathbb{R}^m)$, by lemma 3.3, we can find constants c^i_j such that $|D_j\varphi^i(x)| < c^i_j$ and

$$|\Phi(\xi)| > \max\{|\alpha|, |\beta|\} \quad \forall \xi \in \mathbb{R}^{m\times n}, \quad |\xi^i_j| = c^i_j, \quad i = 1, \dots, m, \quad j = 1, \dots, n. \tag{4.1}$$

We then define

$$E = \{\xi \in \mathbb{R}^{m\times n} : \Phi(\xi) \in \{\alpha, \beta\},\ |\xi^i_j| \leqslant c^i_j,\ i = 1, \dots, m,\ j = 1, \dots, n\}.$$

As before, we only need to verify that the sets E and $\operatorname{Rco} E$ have the approximation property.

Let

$$E_\delta = \{\xi \in \mathbb{R}^{m\times n} : \Phi(\xi) \in \{\alpha+\delta, \beta-\delta\},\ |\xi^i_j| \leqslant c^i_j - \delta,\ i = 1,\dots,m,\ j = 1,\dots,n\}.$$

We first observe that, by continuity, it follows from (4.1) that

$$|\Phi(\xi)| > \max\{|\alpha+\delta|, |\beta-\delta|\} \quad \forall \xi \in \mathbb{R}^{m\times n}, \quad |\xi^i_j| = c^i_j - \delta \quad \forall (i,j).$$

We can then apply theorem 3.5 to find

$$\operatorname{Rco} E_\delta = \{\xi \in \mathbb{R}^{m\times n} : \Phi(\xi) \in [\alpha+\delta, \beta-\delta],\ |\xi^i_j| \leqslant c^i_j - \delta,\ i = 1,\dots,m,\ j = 1,\dots,n\}.$$

It immediately follows that the first and third conditions of definition 2.2 are verified. It therefore remains to check the second one.

We proceed by contradiction and assume that there exist $\varepsilon > 0$ and a sequence $\eta_n \in E_{1/n}$ with $\operatorname{dist}(\eta_n, E) > \varepsilon$. Since $|(\eta_n)^i_j| \leqslant c^i_j$, we can extract a convergent subsequence, still denoted η_n, and $\eta \in E$ such that $\eta_n \to \eta$, which is at odds with $\operatorname{dist}(\eta_n, E) > \varepsilon$.

We can therefore invoke theorem 2.3 to conclude the proof. □

5. Existence of minimizers

We consider in this section the minimization problem

$$\inf\biggl\{\int_\Omega g(\Phi(Du(x)))\,\mathrm{d}x : u \in \varphi + W^{1,\infty}_0(\Omega;\mathbb{R}^m)\biggr\}, \tag{P}$$

where Ω is a bounded open set of $\mathbb{R}^n$, $\varphi \in W^{1,\infty}(\Omega;\mathbb{R}^m)$ and

(i) $g : \mathbb{R} \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is a lower-semicontinuous non-convex function;

(ii) $\Phi : \mathbb{R}^{m\times n} \to \mathbb{R}$ is quasi-affine and non-constant.

We recall that, in particular, we can have, when $m = n$, $\Phi(\xi) = \det \xi$.

The existence result that we give for problem (P) is based on the assumption that the relaxed problem

$$\inf\biggl\{\int_\Omega Cg(\Phi(Du(x)))\,\mathrm{d}x : u \in \varphi + W^{1,\infty}_0(\Omega;\mathbb{R}^m)\biggr\}, \tag{QP}$$

where Cg is the convex envelope of g, has piecewise C^1 solutions. If φ is affine, this is trivial, since $\bar{u} = \varphi$ is then a solution of (QP). When φ is not affine, the only result available is [6], valid for $m = n$ and $\Phi(\xi) = \det \xi$.

The existence result is the following.

THEOREM 5.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $g : \mathbb{R} \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ a lower-semicontinuous function such that*

$$\lim_{|t|\to+\infty} \frac{g(t)}{|t|} = +\infty \tag{5.1}$$

and $\varphi \in W^{1,\infty}(\Omega;\mathbb{R}^m)$. If (QP) has a solution $u_0 \in C^1_{\text{piec}}(\bar{\Omega};\mathbb{R}^m)$, then there exists $\bar{u} \in \varphi + W^{1,\infty}_0(\Omega;\mathbb{R}^m)$, a solution of (P).

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Proof. Let

$$K = \{t \in \mathbb{R} : Cg(t) < g(t)\}.$$

The assumptions on g ensure that K is open and that it can be written as a countable union of disjoint bounded intervals,

$$K = \bigcup_{j \in \mathbb{N}} (\alpha_j, \beta_j).$$

Moreover, on every $[\alpha_j, \beta_j]$, the function Cg is affine, i.e.

$$Cg(t) = a_j + b_j t, \quad t \in [\alpha_j, \beta_j]. \tag{5.2}$$

We then let

$$\begin{aligned} \Omega_0 &= \{x \in \Omega : g(\Phi(Du_0(x))) = Cg(\Phi(Du_0(x)))\}, \\ \Omega_j &= \{x \in \Omega : \Phi(Du_0(x)) \in (\alpha_j, \beta_j)\}, \quad j = 1, 2, \dots \end{aligned}$$

Since u_0 is piecewise C^1, we find that the sets Ω_j, $j = 1, 2, \dots$, are open.

For every $j = 1, 2, \dots$ such that $\Omega_j \neq \emptyset$, we apply theorem 1.4, with $\varphi = u_0 \in C^1_{\mathrm{piec}}(\bar{\Omega}_j; \mathbb{R}^m)$. In this way, we obtain the existence of $u_j \in u_0 + W_0^{1,\infty}(\Omega_j; \mathbb{R}^m)$ such that

$$\Phi(Du_j) \in \{\alpha_j, \beta_j\} \quad \text{a.e. in } \Omega_j.$$

If we define

$$\bar{u} = \begin{cases} u_0 & \text{in } \Omega_0, \\ u_j & \text{in } \Omega_j, \ j \in \mathbb{N}, \end{cases}$$

we have

$$g(\Phi(D\bar{u})) = Cg(\Phi(D\bar{u})) \quad \text{a.e. in } \Omega. \tag{5.3}$$

We claim that $\bar{u}$ is a solution of (P). Indeed, we have $\bar{u} \in \varphi + W_0^{1,\infty}(\Omega; \mathbb{R}^m)$. Moreover, appealing to (5.2), (5.3) and proposition 2.5, we obtain

$$\begin{aligned} \int_\Omega g(\Phi(D\bar{u}(x)))\,\mathrm{d}x &= \int_\Omega Cg(\Phi(D\bar{u}(x)))\,\mathrm{d}x \\ &= \sum_{j=0}^{\infty} \int_{\Omega_j} Cg(\Phi(Du_j(x)))\,\mathrm{d}x \\ &= \int_{\Omega_0} Cg(\Phi(Du_0(x)))\,\mathrm{d}x + \sum_{j=1}^{\infty} \int_{\Omega_j} (a_j + b_j \Phi(Du_j(x)))\,\mathrm{d}x \\ &= \int_{\Omega_0} Cg(\Phi(Du_0(x)))\,\mathrm{d}x + \sum_{j=1}^{\infty} \int_{\Omega_j} (a_j + b_j \Phi(Du_0(x)))\,\mathrm{d}x \\ &= \int_\Omega Cg(\Phi(Du_0(x)))\,\mathrm{d}x. \end{aligned}$$

Finally, using the fact that u_0 is a solution of (QP) and $\inf(QP) \leqslant \inf(P)$, we obtain that $\bar{u}$ is a solution of (P). $\square$

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